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Counting Curves with Modular Forms

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Abstract

We consider the type IIA string compactified on the Calabi-Yau space given by a degree 12 hypersurface in the weighted projective space $\mathbf{P}^4_{(1,1,2,2,6)}$. We express the prepotential of the low-energy effective supergravity theory in terms of a set of functions that transform covariantly under $PSL(2,\mathbb{Z})$ modular transformations. These functions are then determined by monodromy properties, by singularities at the massless monopole point of the moduli space, and by $S \leftrightarrow T$ exchange symmetry.

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1. Introduction

Heterotic/Type II string duality has focused attention on the special Kähler geometry of vectormultiplets as a means of defining some nonperturbative effects in the heterotic string [1][2].

In this note we describe one means for obtaining the prepotential for IIA vectormultiplet geometry using properties of modular forms. In particular we focus on the model described in [2] based on IIA compactification on a manifold $X(1,1,2,2,6)$ with $(h^{1,1},h^{2,1})=(2,128)$. In this example the Kähler cone has two coordinates S,T and the special Kähler coordinate S can be identified with the heterotic dilaton [2][3]. Therefore, on the heterotic side, nonrenormalization theorems show that the (inhomogeneous) vectormultiplet prepotential has the general form:

$$\mathcal{F}(S,T)=ST^2+f_0(T)+\sum_{k=1}^{\infty}f_k(T)e^{2\pi ikS} \tag{1.1}$$

while on the type IIA side we have:

$$\mathcal{F}(S,T)=ST^2+r(T)+\frac{1}{(2\pi i)^3}\sum_{j,k}n_{j,k}Li_3(e^{2\pi i(jT+kS)}) \tag{1.2}$$

where Li_3 is the trilogarithm, $n_{j,k}$ counts rational curves in $X(1,1,2,2,6)$ and $r(T)$ is a cubic polynomial.

The prepotential (1.2) in this model has been computed in [4][5] using mirror symmetry. In this paper we suggest another method - based on monodromy and singularity structure - by which one can determine the prepotential. The procedure may be viewed as a generalization of the method used in [6][7][8] to determine the one-loop prepotential $f_0(T)$ for this model.

In brief, we use the nonperturbative monodromy of the special Kähler periods determined in [9] to find a set of transformation laws for the prepotential. The monodromy group of [9] is a discrete subgroup of $Sp(6;\mathbb{Z})$. It acts on the Kähler cone, and, in the limit $q_2=e^{2\pi iS}\to 0$ the action reduces to the standard Möbius action of $PSL(2,\mathbb{Z})$ on T. Therefore, one may expect the functions $f_k(T)$ to be related to modular forms for $PSL(2,\mathbb{Z})$. We find that this is indeed so. More precisely, we can make an upper triangular transformation of differential polynomials from $f_k(T)$ to a new set of functions $h_k(T)$ which are modular forms. This transformation is summarized in equations (3.1) – (3.4) below. The relation of the prepotential to

modular forms has also been investigated in [10].

Assuming the singularity structure at $T = i$ implied by the connection [2][9] to the Seiberg-Witten massless monopole singularity we find that $h_k(T)$ can be written in terms of polynomials of Eisenstein series. Finally, using the $T \leftrightarrow S$ symmetry implied by $n_{j,k} = n_{j,j-k}$ [5] we find that the polynomial in Eisenstein series is uniquely determined. The $T \leftrightarrow S$ symmetry has been the subject of much recent discussion [11][12].

We must emphasize that the crucial "upper triangular" transformation was discovered "experimentally" using a computer and we have not proved it analytically, although it has been checked extensively.

We hope that this work might offer an alternative method to the standard mirror symmetry techniques for counting rational curves in (some) Calabi-Yau manifolds, which might be of interest in multiparameter examples. A problem with this approach, though, is that one would first have to determine the monodromies. More speculatively, our methods should help to determine root supermultiplicities of generalized Kac-Moody superalgebras [13][14][15][16].

2. The monodromy action

We will consider the type IIA string compactified on the Calabi-Yau threefold given by a degree 12 hypersurface in the weighted projective space $\mathbf{P}^4_{(1,1,2,2,6)}$ discussed in [2][4][5]. The degrees of freedom of the low-energy supergravity theory are described by three vector superfields X^0, X^1 and X^2, corresponding to the graviphoton and two abelian Yang-Mills multiplets respectively. Their dynamics are governed by a holomorphic prepotential $F(X^0, X^1, X^2)$. To get the correct number of propagating fields, F must be homogeneous of degree two in the X^i [17]. It is convenient to introduce the inhomogeneous special coordinates S and T, which are defined in terms of the homogeneous coordinates X^0, X^1 and X^2 as $S = X^1/X^0$ and $T = X^2/X^0$. The prepotential can then be written as

$$F(X^0, X^1, X^2) = (X^0)^2 \left(ST^2 + f(S,T)\right), \tag{2.1}$$

where the first term arises at tree-level and the second term encodes all (perturbative and non-perturbative) quantum corrections.

We define the periods F_i for $i = 0, 1, 2$ as $F_i = \frac{\partial}{\partial X^i} F$, i.e.

$$\begin{aligned} F_0 &= X^0 \left(-ST^2 + 2f - S\frac{\partial f}{\partial S} - T\frac{\partial f}{\partial T} \right) \\ F_1 &= X^0 \left(T^2 + \frac{\partial f}{\partial S} \right) \\ F_2 &= X^0 \left(2ST + \frac{\partial f}{\partial T} \right), \end{aligned} \tag{2.2}$$

and assemble the homogeneous coordinates and the periods in a period vector Π given by $\Pi = \begin{pmatrix} X^0 & X^1 & X^2 & F_0 & F_1 & F_2 \end{pmatrix}^T$. As we encircle a singular divisor in the moduli space, the period vector Π is acted on by multiplication by an element of the monodromy group. This group is a subgroup of $Sp(6, \mathbf{Z})$ generated by three elements S_1, T_1 and T_2 [9], which in our basis are given by

$$\begin{aligned} S_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ T_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ -5 & -1 & -4 & 1 & 0 & -1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{pmatrix} \\ T_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

The monodromies under T_1, T_2 can be deduced, on the type II side, from the monodromy of the mirror manifold at ∞ in complex structure space. The monodromy under S_1 must then be deduced, on the type II side, by solving Picard-Fuchs equations. Alternatively, assuming string/string duality, it may be deduced, on the heterotic side, from the one-loop approximation to $f(S, T)$.

We will now determine the action of the monodromy group on the special coordinates S and T and the function f appearing in the prepotential (2.1). It follows from the above

that the T_2 transformation acts as

$$\begin{aligned} S &\to S+1 \\ T &\to T \\ f &\to f. \end{aligned} \tag{2.4}$$

This means that f may be expanded in powers of $q_2 = \exp 2\pi i S$ [18], i.e.

$$f(S,T) = \sum_{k=0}^{\infty} q_2^k f_k(T) \tag{2.5}$$

for some functions f_k. The T_1 transformation acts as

$$\begin{aligned} S &\to S \\ T &\to T+1 \\ f &\to f + 2T^2 - \frac{5}{2}. \end{aligned} \tag{2.6}$$

In terms of the functions f_k in (2.5), this means that

$$f_0(T+1) = f_0(T) + 2T^2 - \frac{5}{2} \tag{2.7}$$

and

$$f_k(T+1) = f_k(T), \quad k > 0. \tag{2.8}$$

The consequences of the S_1 transformation are less straightforward to extract. It acts as

$$\begin{aligned} S \to \tilde{S} &= 1 + \frac{S + T^{-2} - 2T^{-2} f + S T^{-2} \frac{\partial f}{\partial S} + T^{-1} \frac{\partial f}{\partial T}}{1 + T^{-2} \frac{\partial f}{\partial S}} \\ T \to \tilde{T} &= -\frac{T^{-1}}{1 + T^{-2} \frac{\partial f}{\partial S}} \end{aligned} \tag{2.9}$$

and

$$f \to \frac{T^{-4} f - \frac{1}{2} T^{-4} - T^{-2} - \frac{1}{2} + \ldots}{\left(1 + T^{-2} \frac{\partial f}{\partial S}\right)^3}, \tag{2.10}$$

where the omitted terms are proportional to powers of $\frac{\partial f}{\partial S}$. Inserting (2.5) and (2.9) in (2.10) and taking the $S \to i\infty$ limit, we find that

$$f_0(-1/T) = T^{-4} f_0(T) - \frac{1}{2} T^{-4} - T^{-2} - \frac{1}{2}. \tag{2.11}$$

The transformation properties of the functions f_k for $k > 0$ are most easily deduced by considering $\frac{\partial f}{\partial S}$, which transforms under the S_1 transformation as

$$\frac{\partial f}{\partial S} \to \frac{T^{-4} \frac{\partial f}{\partial S}}{\left(1 + T^{-2} \frac{\partial f}{\partial S}\right)^2}. \tag{2.12}$$

Inserting (2.5) and (2.9) in (2.12), we get

$$\sum_{k=1}^{\infty} 2\pi i k f_k(\tilde{T}) \exp 2\pi i k \tilde{S} = \frac{T^{-4} \sum_{k=1}^{\infty} 2\pi i k f_k(T) \exp 2\pi i S}{\left(1 + T^{-2} \sum_{k=1}^{\infty} 2\pi i k f_k(T) \exp 2\pi i S\right)^2}, \tag{2.13}$$

where $\tilde{S}$ and $\tilde{T}$ are given in (2.9). By expanding this equation in powers of $q_2 = \exp 2\pi i S$ and expanding the functions f_k on the left hand side in a Taylor series around $-1/T$, we can recursively determine the transformation laws of these functions. For example, for f_1 and f_2 we get

$$\begin{aligned}
f_1(-1/T) &= \exp 2\pi i \left(-T^{-2} + 2T^{-2} f_0(T) - T^{-1} f_0^{(1)}(T)\right) T^{-4} f_1(T) \\
f_2(-1/T) &= \exp 4\pi i \left(-T^{-2} + 2T^{-2} f_0(T) - T^{-1} f_0^{(1)}(T)\right) \\
&\qquad \times \Big(T^{-4} f_2(T) \\
&\qquad\quad + T^{-5}(-2\pi i f_1(T) f_1^{(1)}(T)) \\
&\qquad\quad + T^{-6}(4\pi i f_1(T) f_1(T) - 2\pi^2 f_1(T) f_1(T) f_0^{(2)}(T)) \\
&\qquad\quad + T^{-7} 4\pi^2 f_1(T) f_1(T) f_0^{(1)}(T) \\
&\qquad\quad + T^{-8}(2\pi^2 f_1(T) f_1(T) - 4\pi^2 f_0(T) f_1(T) f_1(T))\Big) \\
&\ \ \cdots
\end{aligned} \tag{2.14}$$

where superscripts in parenthesis indicate derivatives with respect to T.

3. Transformation to modular forms

The transformation laws (2.7), (2.8), (2.11) and (2.14) indicate that the functions f_k are closely related to modular forms. Indeed, it follows from (2.7) and (2.11) that $f_0^{(5)}$ is a modular form of weight 6 [6][19].

To see how the f_k for $k > 0$ are related to modular forms, we first introduce a new formal expansion parameter q (distinct from $q_1 = \exp 2\pi i T$ and $q_2 = \exp 2\pi i S$) and a function h with an expansion of the form

$$h = \sum_{k=0}^{\infty} q^k h_k(T). \tag{3.1}$$

Next, we define a set of functions $g_{(m)}$ for $m = 0, 1, \ldots$ recursively by

$$
\begin{aligned}
g_{(0)} &= \frac{\partial}{\partial S} f \\
g_{(m+1)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{mn+1} \left(\frac{(2m-1)(-1)^m}{6(2m)!} h^{(2m+2)} \right)^n \frac{\partial^n}{\partial S^n} (g_{(m)})^{mn+1},
\end{aligned}
\tag{3.2}
$$

where again superscripts in parenthesis denote derivatives with respect to T. Finally, we *define* the relationship between the functions h and f to be given by the equation

$$
h = \lim_{m \to \infty} \left(\frac{\partial}{\partial S} \right)^{-1} g_{(m)} \Bigg|_{q_2 = q}, \tag{3.3}
$$

where the integration constant arising from the $\left(\frac{\partial}{\partial S}\right)^{-1}$ operator is fixed by the requirement that the q independent term of the right-hand side of (3.3) equal $f_0(T)$. Our conjecture is now that the $h_k(T)$ for $k > 0$ transform covariantly with weight -4 under the modular group:

$$
\begin{aligned}
h_k(T+1) &= \exp\left(\frac{2\pi i k}{3} \right) h_k(T) \\
h_k(-1/T) &= \exp\left(\frac{2\pi i k}{3} \right) T^{-4} h_k(T).
\end{aligned}
\tag{3.4}
$$

We have no analytic proof of this statement, but it has been checked by computer up to $k = 7$.

The explicit formulae for the $h_k(T)$ can be obtained as follows. We begin by expanding (3.2) to get:

$$
\begin{aligned}
g_{(0)} &= \sum_{k=1}^{\infty} q_2^k 2\pi i k f_k \\
g_{(1)} &= \sum_{k=1}^{\infty} q_2^k \exp\left(-\frac{\pi i k}{3} h^{(2)} \right) 2\pi i k f_k \\
g_{(2)} &= \sum_{k=1}^{\infty} q_2^k \exp\left(-\frac{\pi i k}{3} h^{(2)} \right) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(-\frac{\pi i k}{6} h^{(4)} \right)^n \\
&\quad \times \sum_{k_0, \ldots, k_n = 1}^{\infty} \delta_{k_0 + \ldots + k_n, k} \prod_{i=0}^{n} 2\pi i k_i f_{k_i} \\
&\ldots
\end{aligned}
\tag{3.5}
$$

Then, inserting (3.1) in (3.3) and solving recursively for the f_k in terms of the h_k, we get

$$
\begin{aligned}
f_0 &= h_0 \\
f_1 &= \exp\left(\frac{\pi i}{3} h_0^{(2)}\right) h_1 \\
f_2 &= \exp\left(\frac{2\pi i}{3} h_0^{(2)}\right) \left(h_2 + \frac{\pi i}{3} h_1 h_1^{(2)} - \frac{\pi^2}{6} h_1 h_1 h_0^{(4)}\right) \\
f_3 &= \exp\left(\frac{3\pi i}{3} h_0^{(2)}\right) \left(h_3 + \frac{2\pi i}{3} h_2 h_1^{(2)} - \frac{\pi^2}{6} h_1 h_1^{(2)} h_1^{(2)} + \frac{\pi i}{3} h_1 h_2^{(2)}\right. \\
&\quad - \frac{2\pi^2}{3} h_1 h_2 h_0^{(4)} - \frac{2\pi^3 i}{9} h_1 h_1 h_1^{(2)} h_0^{(4)} + \frac{\pi^4}{18} h_1 h_1 h_1 h_0^{(4)} h_0^{(4)} \\
&\quad \left. - \frac{\pi^2}{6} h_1 h_1 h_1^{(4)} + \frac{\pi^3 i}{18} h_1 h_1 h_1 h_0^{(6)}\right)
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
f_4 = \exp\left(\frac{4\pi i}{3} h_0^{(2)}\right) &\left(h_4 + \pi i h_3 h_1^{(2)} - \frac{4\pi^2}{9} h_2 h_1^{(2)} h_1^{(2)} - \frac{8\pi^3 i}{81} h_1 h_1^{(2)} h_1^{(2)} h_1^{(2)}\right. \\
&+ \frac{2\pi i}{3} h_2 h_2^{(2)} - \frac{4\pi^2}{9} h_1 h_1^{(2)} h_2^{(2)} + \frac{\pi i}{3} h_1 h_3^{(2)} - \frac{2\pi^2}{3} h_2 h_2 h_0^{(4)} \\
&- \pi^2 h_1 h_3 h_0^{(4)} - \frac{10\pi^3 i}{9} h_1 h_2 h_1^{(2)} h_0^{(4)} + \frac{13\pi^4}{54} h_1 h_1 h_1^{(2)} h_1^{(2)} h_0^{(4)} \\
&- \frac{\pi^3 i}{3} h_1 h_1 h_2^{(2)} h_0^{(4)} + \frac{4\pi^4}{9} h_1 h_1 h_2 h_0^{(4)} h_0^{(4)} \\
&+ \frac{4\pi^5 i}{27} h_1 h_1 h_1 h_1^{(2)} h_0^{(4)} h_0^{(4)} - \frac{2\pi^6}{81} h_1 h_1 h_1 h_1 h_0^{(4)} h_0^{(4)} h_0^{(4)} \\
&- \frac{2\pi^2}{3} h_1 h_2 h_1^{(4)} - \frac{5\pi^3 i}{18} h_1 h_1 h_1^{(2)} h_1^{(4)} + \frac{\pi^4}{6} h_1 h_1 h_1 h_0^{(4)} h_1^{(4)} \\
&- \frac{\pi^2}{6} h_1 h_1 h_2^{(4)} + \frac{\pi^3 i}{3} h_1 h_1 h_2 h_0^{(6)} - \frac{\pi^4}{9} h_1 h_1 h_1 h_1^{(2)} h_0^{(6)} \\
&\left. - \frac{\pi^5 i}{18} h_1 h_1 h_1 h_1 h_0^{(4)} h_0^{(6)} + \frac{\pi^3 i}{18} h_1 h_1 h_1 h_1^{(6)} + \frac{\pi^4}{216} h_1 h_1 h_1 h_1 h_0^{(8)}\right)
\end{aligned}
$$

$\cdots$

(To determine f_k for a given k, it is in fact enough to let $m = k$ in (3.3) rather than taking the limit $m \to \infty$.) Assigning a formal weight $-4+2n$ and a charge k to $h_k^{(n)}$, we see that the general structure is that f_k equals $\exp\left(\frac{k\pi i}{3} h_0^{(2)}\right)$ times a differential polynomial in h of formal weight -4 and charge k involving only even derivatives. The 'upper triangular'

structure of these equations makes them easy to invert:

$$
\begin{aligned}
h_0 &= f_0 \\
h_1 &= \exp\left(-\frac{\pi i}{3} f_0^{(2)}\right) f_1 \\
h_2 &= \exp\left(-\frac{2\pi i}{3} f_0^{(2)}\right) \\
&\quad \times \left(f_2 - \frac{\pi i}{3} f_1^{(2)} f_1 - \frac{2\pi^2}{9} f_0^{(3)} f_1^{(1)} f_1 + \frac{\pi^2}{18} f_0^{(4)} f_1 f_1 + \frac{\pi^3 i}{27} f_0^{(3)} f_0^{(3)} f_1 f_1\right) \\
&\ldots
\end{aligned}
\tag{3.7}
$$

4. The singularity structure

Next, we need some facts about the singularities of the functions f_k. These occur at $T = i$, where the gauge group is enlarged and additional multiplets become massless. As $T \to i$, $f_0^{(2)}(T)$ diverges as $-\frac{8}{2\pi i} \log(T - i)$ [6][19][20]. Furthermore, to recover the results of [21] in the limit when the string tension becomes large, $f_k(T)$ for $k > 0$ must have a pole of order $4k - 2$ at $T = i$ [9]. Finally, it follows from (2.7) that $f_0^{(2)}$ diverges as $4T$ as $T \to i\infty$. From the general form of the functions h_k (3.7), we then conclude that their singularities are given by

$$
\begin{aligned}
h_k(T) &\sim (T - i)^{2-8k/3}, \qquad T \to i \\
h_k(T) &\sim q_1^{-2k/3}, \qquad T \to i\infty.
\end{aligned}
\tag{4.1}
$$

From the fact that $f_0^{(5)}$ is a modular form of weight 6 and the above singularity structure, it can be determined [7][8] as

$$
f_0^{(5)} = (2\pi i)^2 \frac{18E_4^6 - 23E_4^3 E_6^2 + 5E_6^4}{9E_6^3},
\tag{4.2}
$$

where the Eisenstein series of weight 4 and 6 are defined as

$$
\begin{aligned}
E_4(T) &= 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q_1^j}{1 - q_1^j} \\
E_6(T) &= 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q_1^j}{1 - q_1^j}
\end{aligned}
\tag{4.3}
$$

for $q_1 = \exp 2\pi i T$. They have simple zeros at $T = \exp(\pi i/3)$ and at $T = i$ respectively. Integrating $f_0^{(5)}$ five times gives f_0; all integration constants except one are determined by imposing (2.7):

$$f_0(T) = \text{constant} - \frac{13}{6}T - T^2 + \frac{2}{3}T^3 + \mathcal{O}(q_1). \tag{4.4}$$

From (3.4) and (4.1), we conclude that for $k > 0$

$$h_k = (2\pi i)^{-3} \frac{P_{24k-16}}{\eta^{16k} E_6^{8k/3-2}}, \tag{4.5}$$

where the η invariant is defined as

$$\eta(T) = q_1^{1/24} \prod_{j=1}^{\infty} \left(1 - q_1^j\right) \tag{4.6}$$

and P_{24k-16} is a holomorphic modular form of weight $24k - 16$. As such, P_{24k-16} must be of the form

$$P_{24k-16} = \sum_{n=1}^{2k-1} p_{k,n} E_4^{3n-1} E_6^{4k-2n-2}, \tag{4.7}$$

for some constants $p_{k,1}, \ldots, p_{k,2k-1}$.

5. The exchange symmetry

To determine the forms P_{24k-16} in (4.5), we consider the Yukawa coupling $\left(\frac{\partial}{\partial T}\right)^3 f$, which may be written in the form [22]

$$\frac{\partial^3}{\partial T^3} f = 4 + \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} j^3 n_{j,k} \frac{q_1^j q_2^k}{1 - q_1^j q_2^k}. \tag{5.1}$$

For fixed $k > 0$, the $S \leftrightarrow T$ exchange symmetry [5][11][12] amounts to $2k - 1$ constraints on the instantonnumbers $n_{j,k}$ in (5.1):

$$n_{j,k} = \begin{cases} 0 & , \quad 1 \le j \le k-1 \\ n_{j,j-k} & , \quad k \le j \le 2k-1 \end{cases}. \tag{5.2}$$

The h_k can now be determined recursively: Given $h_1, \ldots, h_{k-1}$, the constraints (5.2) determine the constants $p_{k,1}, \ldots, p_{k,2k-1}$ in (4.7) and thus, by (4.5), h_k. Explicitly, the first

few h_k's are given by

$$
\begin{aligned}
h_1 &= (2\pi i)^{-3}\eta^{-16}E_6^{-2/3}2E_4^2 \\
h_2 &= (2\pi i)^{-3}2^{-4}3^{-3}\eta^{-32}E_6^{-10/3}\left(-89E_4^8-53E_4^5E_6^2+122E_4^2E_6^4\right) \\
h_3 &= (2\pi i)^{-3}2^{-5}3^{-7}\eta^{-48}E_6^{-6} \\
&\quad\times\left(20367E_4^{14}-38052E_4^{11}E_6^2+18898E_4^8E_6^4-6260E_4^5E_6^6+3895E_4^2E_6^8\right) \\
h_4 &= (2\pi i)^{-3}2^{-13}3^{-10}\eta^{-64}E_6^{-26/3} \\
&\quad\times\left(216412213E_4^{20}-793763223E_4^{17}E_6^2+1110594390E_4^{14}E_6^4\right. \\
&\qquad -711685317E_4^{11}E_6^6+217366407E_4^8E_6^8-18944802E_4^5E_6^{10} \\
&\qquad \left.+5991276E_4^2E_6^{12}\right) \\
&\ldots
\end{aligned}
\tag{5.3}
$$

(We have also calculated h_5, h_6 and h_7.) Inserting this result in (3.6) gives the functions f_k in the prepotential. The instanton expansion (5.1) then gives the instantonnumbers $n_{j,k}$, which count rational curves in the Calabi-Yau manifold:

j	k = 0	k = 1	k = 2 . . .
0	0	2	0
1	2496	2496	0
2	223752	1941264	223752
3	38637504	1327392512	1327392512
4	9100224984	861202986072	2859010142112
5	2557481027520	540194037151104	4247105405354496
6	805628041231176	331025557765003648	5143228729806654496
7	274856132550917568	199399229066445715968	5458385566105678112256
. . .			

These numbers agree with the results of [5].

Thus, we have established the claim that the monodromy, singularity structure, and $S \leftrightarrow T$ exchange symmetry are sufficient to determine the prepotential exactly in the region where the series converges.

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